Introduction to the Theory of Computation

Mathematical Foundations and Preliminaries

Theory of Computation Slide Sets

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The Fundamental Question

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- **Computability Theory:** What is solvable vs. unsolvable? We determine if an algorithm can exist for a given problem (e.g., the Halting Problem).
- **Complexity Theory:** What is computationally easy vs. hard? We classify problems based on resource requirements like time and space (e.g., P vs. NP).

Part 2: Sets and Basics

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- **Subset:** $A \subseteq B$ means every element of A is also in B.
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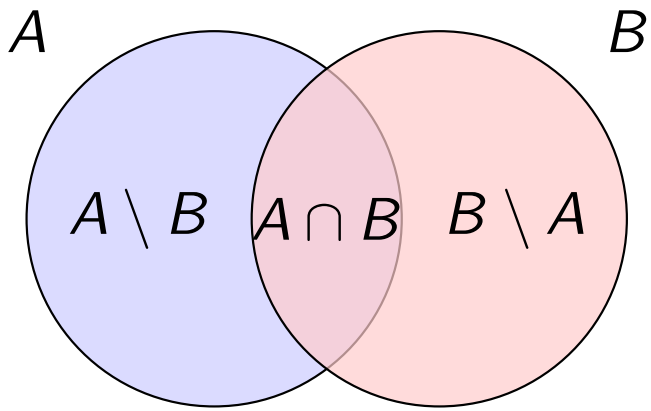
Teacher's Corner: A Common Pitfall

Students frequently confuse the empty set $\emptyset$ with the empty string ε.

- $\emptyset$ is a *set* that contains absolutely nothing: $|\emptyset| = 0$.
- ε is a *string* of length zero.
- The set $\{\varepsilon\}$ is *not* empty! It is a set containing exactly one element (the empty string).

Set Operations and Venn Diagrams

- **Union** ($A \cup B$): Elements in A or B.
- **Intersection** ($A \cap B$): Elements in both.
- **Difference** ($A - B$ or $A \setminus B$): Elements in A but not B.
- **Complement** ($\overline{A}$): Elements not in A (relative to a universal set).



$A \cup B$ is the entire colored region

Power Sets and Cartesian Products

Power Set

The **power set** of S, denoted 2^S (or $\mathcal{P}(S)$), is the set of all subsets of S.
Example: If $S = \{a, b, c\}$, then:

$$2^S = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Note: $|2^S| = 2^{|S|}$.

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Sequences, Tuples, and Cartesian Products

A **sequence** is an ordered list. A finite sequence is a **tuple**. An ordered pair is a 2-tuple.
Cartesian Product ($A \times B$): The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.
Example: $A = \{1, 2\}$, $B = \{x, y, z\}$

$$A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z)\}$$

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Example

Let $f : \mathbb{Z} \to \mathbb{Z}$ be $f(x) = |x|$. The domain is $\mathbb{Z}$ and the codomain is $\mathbb{Z}$. The actual **range** is the set of non-negative integers ($\mathbb{Z}^+$), meaning the function is not *onto* the codomain.

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Quick Check

- $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ is **bijective**.
- $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ is **neither** (fails injective for negative inputs, fails surjective for negative outputs).

Relations and Equivalence

A **relation** is a property whose domain is a set of k-tuples. A binary relation R on a set A is a subset of $A \times A$. We often write aRb instead of $(a, b) \in R$.

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Equivalence Relation

A binary relation R is an **equivalence relation** if it satisfies three conditions:

1. **Reflexivity:** xRx for every x.
2. **Symmetry:** If xRy, then yRx.
3. **Transitivity:** If xRy and yRz, then xRz.

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Equivalence Classes

An equivalence relation partitions a set into disjoint **equivalence classes**. Elements in the same class are deemed "equivalent." (Crucial later for state minimization in automata).

Boolean Logic

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Precedence

Just like arithmetic ($\times$ before $+$), boolean operations have an evaluation order:

NOT precedes AND, and AND precedes OR.

Example: $P \vee Q \wedge \neg R$ is evaluated as $P \vee (Q \wedge (\neg R))$.

Part 4: Graph Theory Basics

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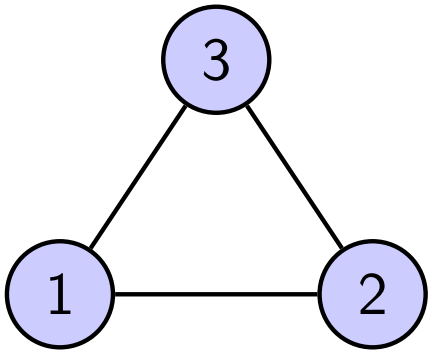
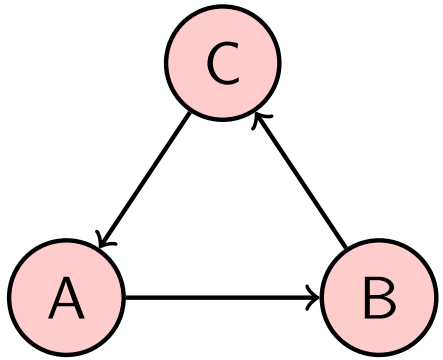
Undirected Graph: Edges have no direction. E consists of unordered pairs $\{u, v\}$.



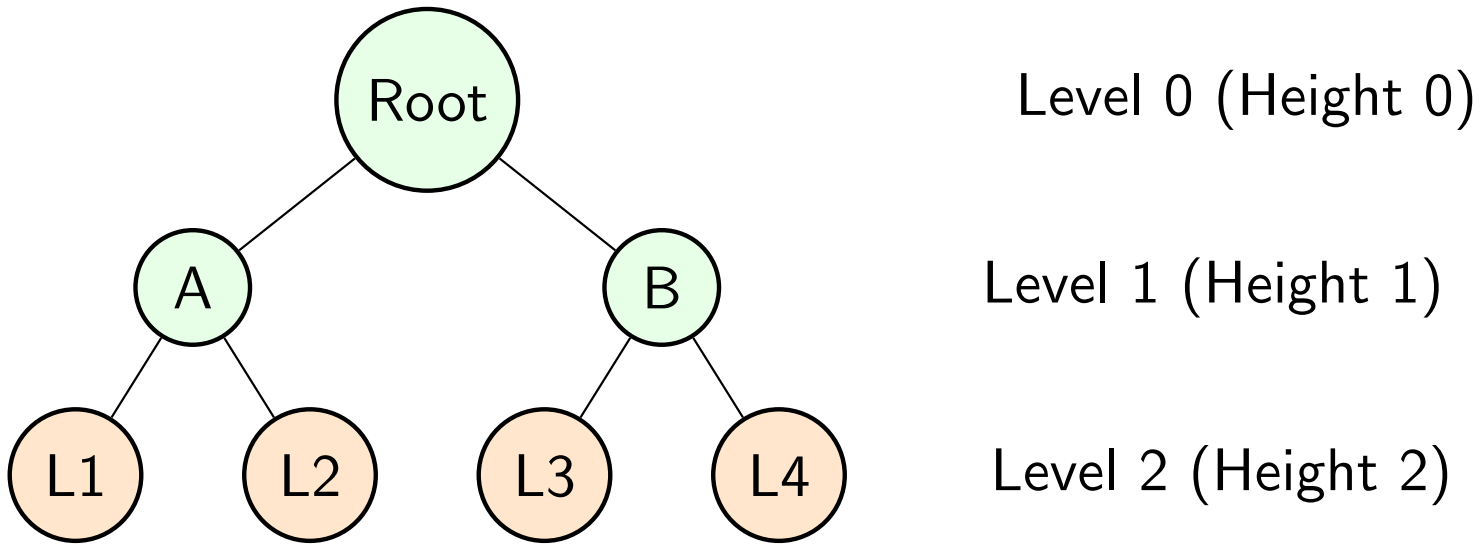
Directed Graph (Digraph): Edges have direction. E consists of ordered pairs (u, v).



A **path** is a sequence of nodes connected by edges. A **cycle** is a path that starts and ends at the same node.

Tree Structures

A **tree** is a connected directed graph with no simple cycles, where one node is designated the **root**, and every other node has exactly one incoming edge (from its **parent**). Nodes with no outgoing edges are **leaves**.



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- **Empty String (ε or λ):** The string of length zero. $|\varepsilon| = 0$.
- **Concatenation:** Appending strings. If $x = 01$ and $y = 11$, $xy = 0111$.
- **Reversal (w^R):** The string written backwards. $(011)^R = 110$.

Languages and the Kleene Star

Kleene Star (Σ^*)

The set of *all possible strings* over an alphabet Σ, including the empty string ε.

Example: If $\Sigma = \{0, 1\}$, then $\Sigma^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \dots\}$.

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A **language** L is simply a subset of Σ^*.

$$L \subseteq \Sigma^*$$

A language can be finite or infinite. Even the empty set $\emptyset$ is a language.

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Pedagogical Note

In computer science, a "language" is just a set of strings. The strings don't need to mean anything! A compiler's job is essentially answering the membership question: *Is the string (source code) a member of the language (valid C++ syntax)?*

Part 6: Grammars as Generators

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Grammar Quadruple

A grammar is a 4-tuple $G = (V, T, S, P)$:

- V: Finite set of **Variables** (Non-terminals).
- T: Finite set of **Terminals** (symbols from the alphabet). $V \cap T = \emptyset$.
- $S \in V$: The **Start Variable**.
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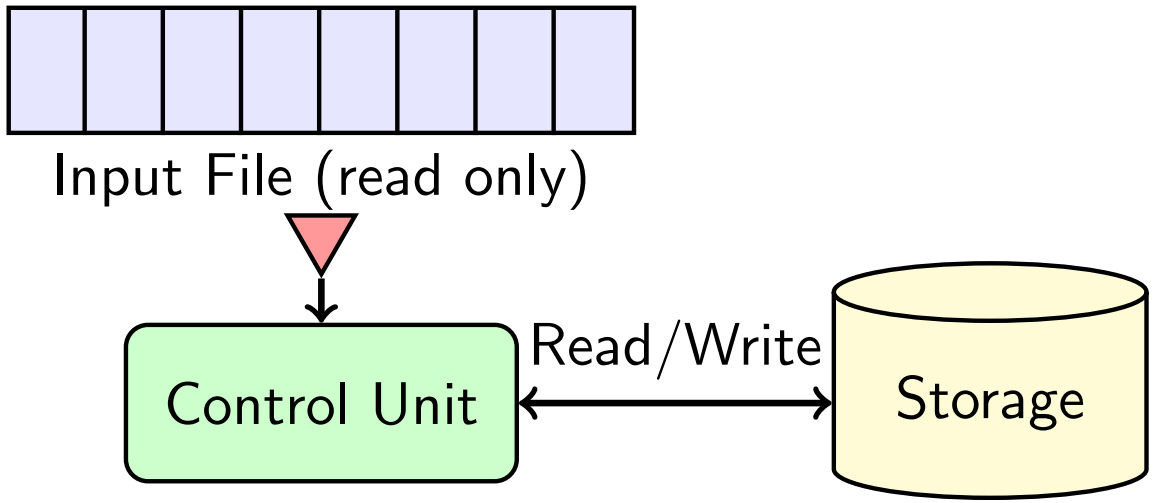
Derivations ($\Rightarrow$): We apply rules from P to replace variables.
Example: $S \to aSb \mid \varepsilon$.
Derivation: $S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$.
The generated language is $L(G) = \{a^n b^n \mid n \geq 0\}$.

Automata as Recognizers

An **automaton** is an abstract model of a digital computer. It processes an input string and decides whether to "Accept" or "Reject".



- **Finite Automata:** No temporary storage.
- **Pushdown Automata:** Storage is a Stack (LIFO).
- **Turing Machines:** Infinite random-access tape.

Part 7: Proof by Contradiction

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Theorem: $\sqrt{2}$ is irrational

Proof: Assume for contradiction that $\sqrt{2}$ is rational.

1. Then $\sqrt{2} = \frac{m}{n}$ where m, n are integers with no common factors.
2. Squaring both sides: $2 = \frac{m^2}{n^2} \implies 2n^2 = m^2$.
3. This means m^2 is even, so m must be even. Let $m = 2k$.
4. Substitute back: $2n^2 = (2k)^2 = 4k^2 \implies n^2 = 2k^2$.
5. This means n^2 is even, so n must be even.
6. If m and n are both even, they share a common factor of 2. This contradicts our assumption! Therefore, $\sqrt{2}$ must be irrational. ■

Proof by Induction

To prove a property $P(n)$ holds for all integers $n \geq 0$:

- **Basis:** Prove $P(0)$ is true.
- **Inductive Step:** Assume $P(k)$ is true (Inductive Hypothesis), and use it to prove $P(k+1)$ is true.

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Theorem: A binary tree of height n has at most 2^n leaves.

Proof: Let $l(n)$ be the max number of leaves at height n.

- **Basis:** For $n = 0$, the tree is just the root. Leaves $= 1$. $2^0 = 1$. True.
- **Inductive Step:** Assume $l(k) \leq 2^k$. A tree of height $k+1$ is formed by taking a tree of height k and adding at most 2 children to each leaf.
- Thus, $l(k+1) \leq 2 \times l(k)$.
- By the hypothesis, $l(k+1) \leq 2 \times 2^k = 2^{k+1}$. ■

Checkpoint Exercises 1: Sets, Relations, and Logic

1. **Sets:** Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.
   What is $(A \cup B) - (A \cap B)$?

2. **Relations:** A relation R on $\mathbb{Z}$ is defined as xRy if $x \leq y$.
   Which equivalence property does this relation *fail*? (Reflexivity, Symmetry, or Transitivity?)

3. **Boolean Logic:** Evaluate the following expression, given $P = 1, Q = 0, R = 1$:
   $$\neg P \vee Q \wedge R$$

Checkpoint Exercises 2: Strings and Grammars

4. **Strings:** Let $\Sigma = \{0, 1\}$. How many strings are in Σ^* of length exactly 3? List them.

5. **Grammars:** Consider the grammar G:

$$S \to aA$$
$$A \to bS \mid \varepsilon$$

Show the derivation sequence for the string *ababa*. What is the general language $L(G)$ described in English?

Checkpoint Exercises 3: Proofs and Graphs

6. **Graphs:** An undirected graph has v vertices and e edges. Prove using a direct argument that the sum of the degrees of all vertices is exactly $2e$.

7. **Induction:** We want to prove by induction that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.
   State explicitly what the **Basis** and the **Inductive Hypothesis** are for this proof.

Hints for Checkpoint Exercises

Slide 16 Hints:

- 1: Find the union first, then find the intersection. Remove the intersection elements from the union.
- 2: Try swapping the variables. If $3 \leq 4$, does $4 \leq 3$?
- 3: Remember precedence: NOT precedes AND, which precedes OR.

Slide 17 Hints:

- 4: The number of strings of length n over an alphabet of size k is k^n.
- 5: Substitute S, then A, then S again until you generate *ababa*, finishing with the ε rule to clear the variable.

Slide 18 Hints:

- 6: Every edge connects exactly two nodes. How many times is an edge counted when you sum the degrees?
- 7: The basis is $n = 1$. The inductive hypothesis assumes the formula holds for an arbitrary k.