Properties of Regular Languages

Closure, Decision Algorithms, and the Pumping Lemma

Computer Science Theory

Slide Set 4

Table of Contents

Learning Outcomes

By the end of this slide set, you should be able to:

- prove closure of the regular languages using automaton constructions and set identities;
- decide membership, emptiness, finiteness, equality, containment, and universality for regular languages;
- state and explain the quantifier structure of the pumping lemma;
- use the pumping lemma to prove selected languages nonregular; and
- recognize invalid pumping-lemma arguments and repair them.

Part 1: Core Closure Properties

Self-Study Header: Operating within the Regular World

A class of languages is **closed** under an operation if applying that operation to members of the class always results in a member of the same class.

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Formally, if L_1 and L_2 are regular languages, the following are also regular:

1. **Union:** $L_1 \cup L_2$
2. **Concatenation:** $L_1 L_2$
3. **Kleene star:** L_1^*
4. **Complement:** $\overline{L_1} = \Sigma^* \setminus L_1$
5. **Intersection:** $L_1 \cap L_2$

Proof by Construction: Complementation

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Because L is regular, a *complete* DFA $M = (Q, \Sigma, \delta, q_0, F)$ recognizes L. Construct $M' = (Q, \Sigma, \delta, q_0, Q \setminus F)$ by swapping the accepting and nonaccepting states.

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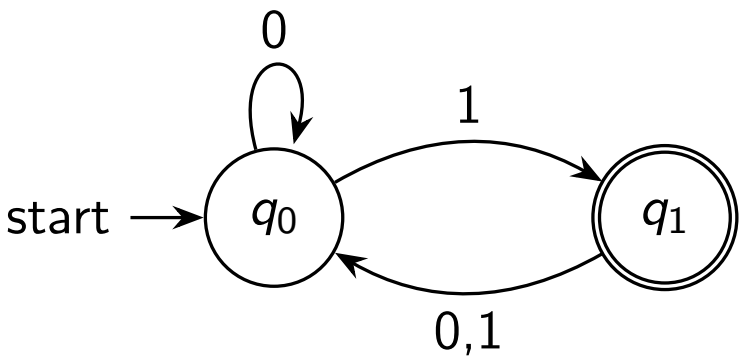
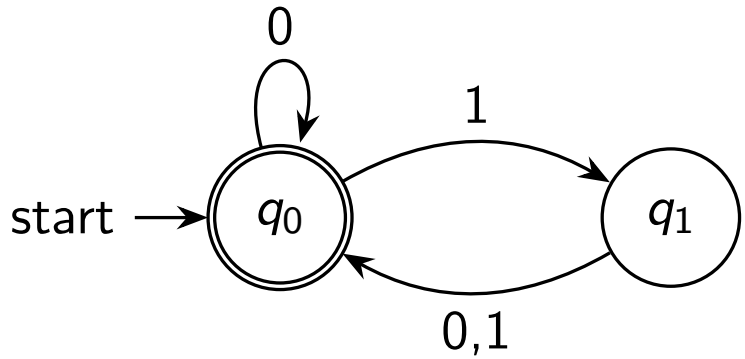


DFA for L



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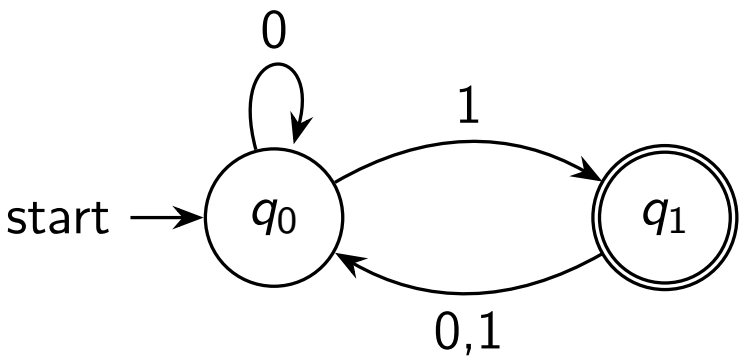

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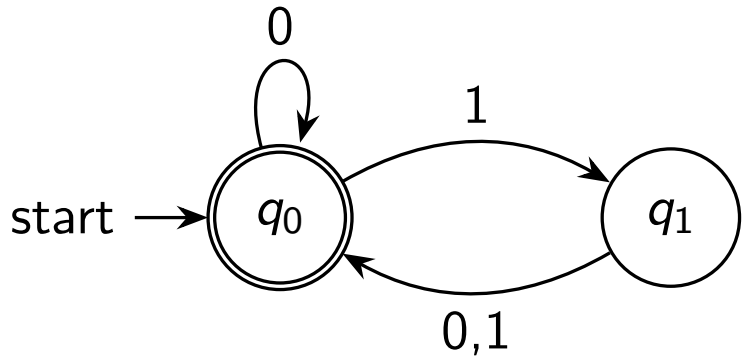

DFA for $\overline{L}$

Crucial detail: Complete the DFA first by adding a sink state if necessary. Merely swapping final states in an NFA does not compute the complement, because acceptance is existential over its possible paths.

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Let $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$. We build a new DFA $M_\cap = (Q, \Sigma, \delta, q_0, F)$ that simulates both machines simultaneously:

- $Q = Q_1 \times Q_2$ (the states are pairs).
- $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$.
- $q_0 = (s_1, s_2)$.
- $F = F_1 \times F_2$ (accept iff **both** machines accept).

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Alternative Proof via De Morgan's Law

Alternatively, $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$. Since regular languages are closed under union and complementation, they must be closed under intersection!

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- **Homomorphism ($h(L)$):** For each transition labeled a, replace the edge by a path spelling $h(a)$; use an ε-transition when $h(a) = \varepsilon$.
- **Right quotient:** $L_1/L_2 = \{x : \text{some } y \in L_2 \text{ satisfies } xy \in L_1\}$. If L_1 and L_2 are regular, then L_1/L_2 is regular.

Closure Proof Toolbox

Choose the representation that makes the operation easiest.

Automaton Constructions

Union/intersection: Use a product DFA; change only the accepting pairs ($\vee$ for union, $\wedge$ for intersection).

Complement: Complete a DFA and replace F by $Q \setminus F$.

Concatenation/star: Connect ε-NFAs using the same wrappers as in Thompson's construction.

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Set Identities Save Work

Difference, symmetric difference, and De Morgan's laws reduce new closure questions to operations already known to preserve regularity.

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Given a standard representation of a regular language L (DFA, NFA, regular expression, or regular grammar), can we algorithmically answer fundamental questions about it?

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1. The Membership Question

Given a string w and a standard representation of L, is $w \in L$?

Algorithm: For a DFA, trace the unique path labeled w from q_0 and test whether it ends in F.
Running time: $O(|w|)$ for a supplied DFA. Converting an NFA, RE, or grammar to a DFA is preprocessing and may increase the representation size substantially.

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Because the standard representations are effectively convertible, a DFA algorithm establishes decidability for all of them, though not necessarily with the same complexity.

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3. The Finiteness Question

Given a regular language L, is L finite or infinite?

Algorithm: Keep the *useful* states that are reachable from q_0 and can reach a final state. Then L is infinite exactly when the subgraph induced by these useful states contains a directed cycle.

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We answer this using the **symmetric difference**:

$$L_3 = (L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2).$$

L_3 contains all strings that are in one language but not the other. $L_1 = L_2$ if and only if $L_3 = \emptyset$.

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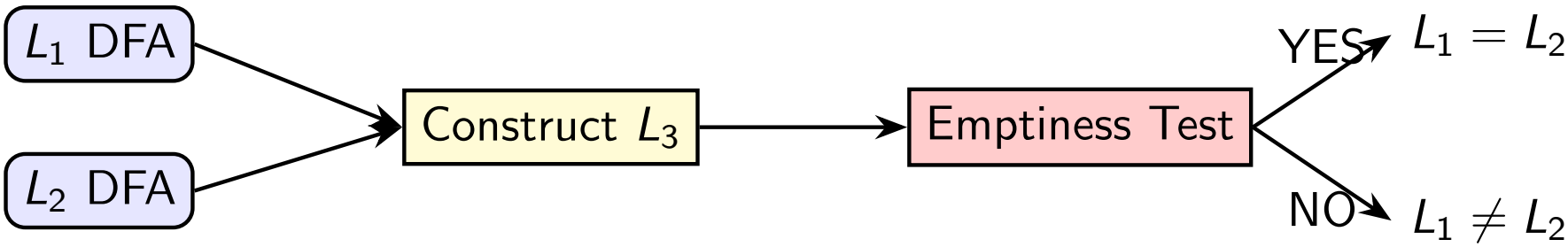

More Decidable Questions

Once emptiness and the product construction are available, several other questions follow immediately.

Reductions to Emptiness

$$L = \Sigma^* \iff \overline{L} = \emptyset \qquad \text{(universality)},$$

$$L_1 \subseteq L_2 \iff L_1 \cap \overline{L_2} = \emptyset \qquad \text{(containment)},$$

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Direct Product Test

For equality, build the product DFA and search for a reachable pair in which exactly one component is accepting. Such a path labels a concrete counterexample string distinguishing the two languages.

Part 3: Concept & The Pigeonhole Principle

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A DFA has finite memory: its current state records everything relevant about the prefix read so far. What happens when a computation is longer than the number of available states?

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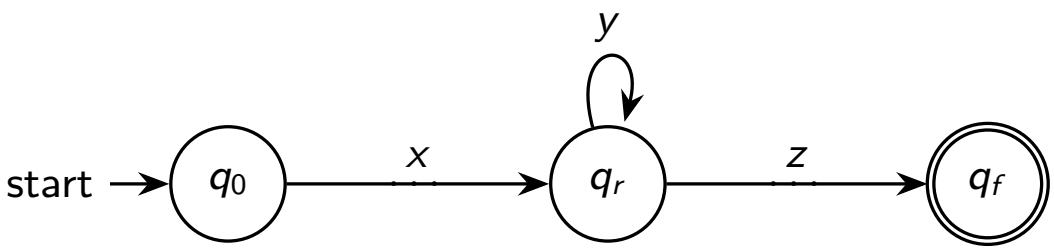
If a DFA with n states reads a string of length at least n, consider the states visited before reading any symbol and after each of the first n symbols. Among these $n + 1$ visits, some state repeats. The intervening nonempty input labels a loop within the first n symbols.



Formal Statement: The Pumping Lemma

Pumping lemma for regular languages

If L is regular, then there exists $p \geq 1$ such that every $s \in L$ with $|s| \geq p$ can be written as $s = xyz$ satisfying:

1. $xy^i z \in L$ for every $i \geq 0$,
2. $|y| > 0$, and
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Breaking it down:

- Condition 1: The same decomposition works for *every* pumping value $i \geq 0$.
- Condition 2: The loop consumes at least one symbol, so pumping actually changes the string when $i \neq 1$.
- Condition 3: The loop lies within the first p symbols.

Why the Pumping Lemma Is True

Let a DFA for L have p states, and let $s \in L$ with $|s| \geq p$. During the first p input symbols, record the $p+1$ visited states:

$$q_0, q_1, \ldots, q_p.$$

Two visits must be equal; choose $0 \leq r < t \leq p$ with $q_r = q_t$. Split s as follows:

$$\underbrace{s_1 \cdots s_r}_{x} \underbrace{s_{r+1} \cdots s_t}_{y} \underbrace{s_{t+1} \cdots s_{|s|}}_{z}.$$

Then $|y| = t - r > 0$ and $|xy| = t \leq p$. Because reading y returns the DFA to the same state, the loop may be traversed any number of times; hence $xy^i z \in L$ for every $i \geq 0$.

Quantifiers and the Lemma's Limitation

The logical order is the heart of a correct nonregularity proof:

$$\exists p\ \forall s\ \exists (x, y, z)\ \forall i \geq 0.$$

To contradict the lemma, reverse the game after assuming regularity:

$$\forall p\ \exists s\ \forall (x, y, z)\ \exists i \geq 0 \quad \text{such that } xy^i z \notin L,$$

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Necessary, Not Sufficient

Every regular language satisfies the pumping lemma, but satisfying the lemma does not by itself prove that a language is regular. Finite languages satisfy it vacuously by choosing p larger than every string length.

The Adversarial Game

The pumping lemma is used **negatively** to prove that a language is not regular. Its quantifiers can be remembered as a four-step game.

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If this strategy works for every valid split, the initial regularity assumption is false.

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Since $k > 0$, the numbers of a's and b's differ. Thus $xy^2z \notin L$, contradicting the pumping lemma. Therefore L is not regular.

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The pumped string has exactly two 1's, separated by $p + 1$ positions. If it were uu, the distance between these two corresponding 1's would equal half the total length, $p + 1 + k/2$, which is impossible for $k > 0$ (and if k is odd, the total length is odd). Hence $xy^2z \notin F$, a contradiction.

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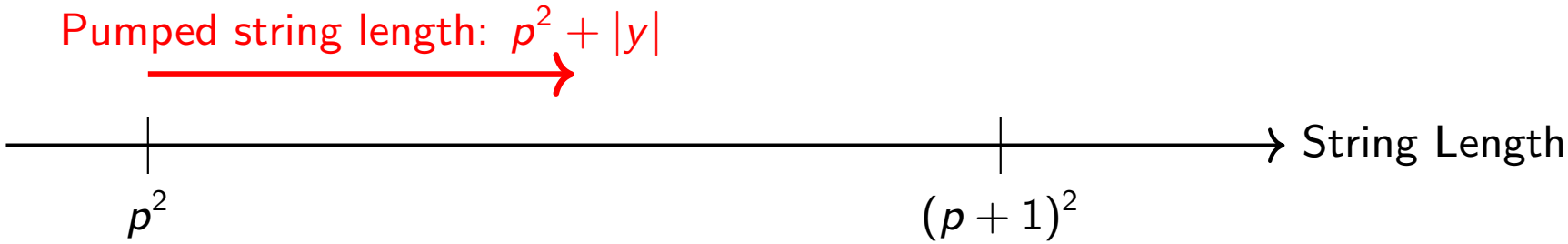

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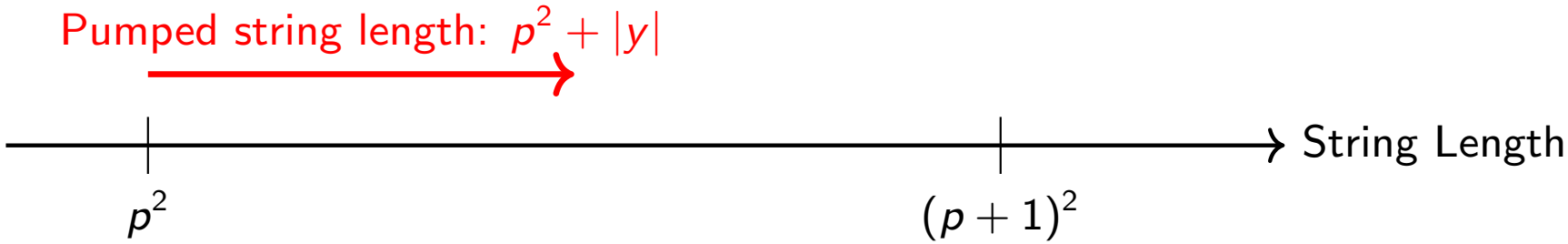

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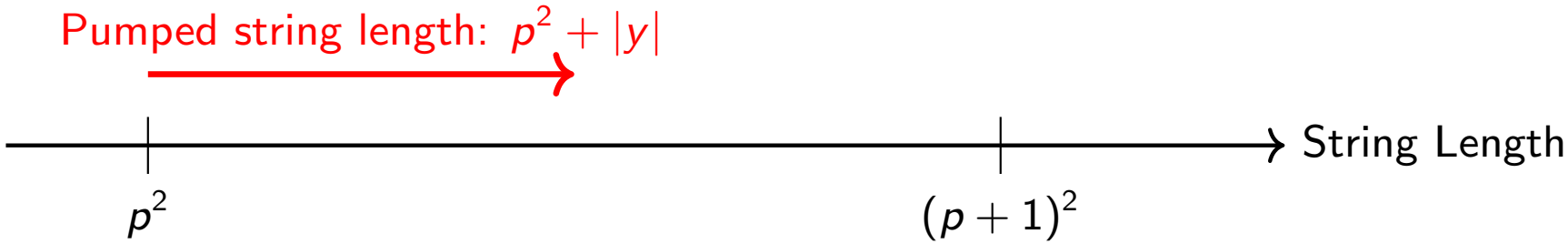
1. Pick $s = a^{p^2}$. Opponent divides $s = xyz$.
2. We pump $i = 2$. The length of xy^2z is $p^2 + |y|$.
3. Since $|y| \leq p$, then $p^2 < p^2 + |y| \leq p^2 + p < p^2 + 2p + 1 = (p+1)^2$.

Example 3: Unary Perfect Squares

Goal: Prove $L = \{a^{n^2} : n \geq 0\}$ is not regular.



Since $|y| \leq p$, the pumped length is at most $p^2 + p$

The next square is $(p+1)^2 = p^2 + 2p + 1$

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The pumped length lies strictly between consecutive squares, so $xy^2z \notin L$. This contradicts the pumping lemma.

Using Closure to Prove Nonregularity

Closure theorems can be used contrapositively.

Reduction Pattern

To prove a language K nonregular:

1. assume K is regular;
2. apply an operation that preserves regularity; and
3. obtain a language already known to be nonregular.

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Example

Let

$$K = \{w \in \{a, b\}^* : |w|_a = |w|_b\}.$$

If K were regular, then its intersection with the regular language a^*b^* would be regular. But

$$K \cap a^*b^* = \{a^n b^n : n \geq 0\},$$

contradicting Example 1. Hence K is not regular.

Checkpoint Exercises

1. **Closure Property Trap:** Given that L_1 is regular, and the union $L_1 \cup L_2$ is regular, does it necessarily follow that L_2 is regular?

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2. **A Surprising Result:** Let D contain the strings over $\{0, 1\}$ having equally many occurrences of 01 and 10 as consecutive substrings. Is D regular?

3. **Choosing s:** If trying to prove $L = \{a^n b^n \mid n \geq 0\}$ is not regular, what is wrong with picking $s = a^2 b^2$ if the opponent's $p = 5$?

Teacher's Corner: Common Pitfalls

Pitfall 1: Picking a specific p

Students sometimes begin by fixing $p = 5$. This is invalid: the pumping length is an unknown constant. The strategy must work for every $p \geq 1$.

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Pitfall 2: Making assumptions about y

Even for $s = a^p b^p$, you may not assume $y = a$. The opponent may choose $y = a^k$ for any $1 \leq k \leq p$. The proof succeeds because pumping *every* such nonempty block changes only the number of a's.

Also verify $s \in L$: for example, $(ab)^p \notin \{a^n b^n : n \geq 0\}$ when $p > 1$, so it cannot be used as the witness string for Example 1.

Hints for Checkpoint Exercises

Hints and Solutions

- **Exercise 1 (Closure Trap):** False. Let $L_1 = \Sigma^*$. Then $L_1 \cup L_2 = \Sigma^*$ for any L_2, including a nonregular language.
- **Exercise 2 (01 and 10 substrings):** Yes. Each change $0 \to 1$ must eventually be balanced by a change $1 \to 0$, except for a possible net change determined by the endpoints. Thus the counts are equal exactly for ε or when the first and last symbols agree; a DFA need only remember the first and current symbols.
- **Exercise 3 (Choosing s):** The witness must satisfy $|s| \geq p$. If $p = 5$, then $|a^2 b^2| = 4$, so the pumping lemma makes no promise about that string.

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Pro-Tip: Pumping Down
Sometimes pumping up ($i = 2$) does not break the defining property. The lemma also requires $xy^0z = xz \in L$, so deleting y is often the cleaner route to a contradiction.