Proof of the Master Method

Theorem (Master Method) Consider the recurrence

$$T(n) = aT(n/b) + f(n), \tag{1}$$

where a, b are constants. Then

(A) If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = O(n^{\log_b a})$.

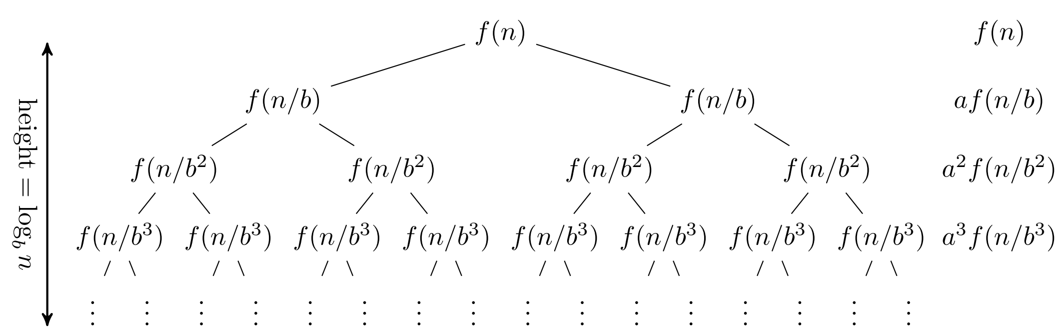
(B) If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.

(C) If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if f satisfies the smoothness condition $af(n/b) \leq cf(n)$ for some constant $c < 1$, then $T(n) = \Theta(f(n))$.

Proof. The solution of the recurrence is

$$T(n) = \sum_{i=0}^{\log_b n} a^i f(n/b^i) + O(n^{\log_b a}). \tag{2}$$

This can be seen by drawing the tree generated by the recurrence (1). The tree has depth $\log_b n$ and branching factor a. There are a^i nodes at level i, each labeled $f(n/b^i)$. The value of $T(n)$ is the sum of the labels of all the nodes of the tree. The sum (2) is obtained by summing the ith level sums. The last term $O(n^{\log_b a})$ is the sum across the leaves, which is $a^{\log_b n} f(1) = n^{\log_b a} f(1)$. The diagram shows the case $a = 2$.



Case (A). From (2), we have

$$T(n) = \sum_{i=0}^{\log_b n} a^i f(n/b^i) + O(n^{\log_b a}) \leq \sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b a - \varepsilon} + O(n^{\log_b a}), \quad (3)$$

and

$$\begin{aligned}
\sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b a - \varepsilon} &= n^{\log_b a - \varepsilon} \sum_{i=0}^{\log_b n} a^i b^{-i \log_b a} b^{i\varepsilon} = n^{\log_b a - \varepsilon} \sum_{i=0}^{\log_b n} a^i a^{-i} b^{i\varepsilon} \\
&= n^{\log_b a - \varepsilon} \sum_{i=0}^{\log_b n} b^{\varepsilon i} = n^{\log_b a - \varepsilon} \frac{b^{\varepsilon(\log_b n + 1)} - 1}{b^\varepsilon - 1} \\
&= n^{\log_b a - \varepsilon} \frac{n^\varepsilon b^\varepsilon - 1}{b^\varepsilon - 1} \leq n^{\log_b a - \varepsilon} \frac{n^\varepsilon b^\varepsilon}{b^\varepsilon - 1} = n^{\log_b a} \frac{b^\varepsilon}{b^\varepsilon - 1} \\
&= O(n^{\log_b a}).
\end{aligned}$$

Combining this with (3), we get $T(n) = O(n^{\log_b a})$.

Case (B). Here we have

$$\begin{aligned}
\sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b a} &= n^{\log_b a} \sum_{i=0}^{\log_b n} a^i b^{-i \log_b a} = n^{\log_b a} \sum_{i=0}^{\log_b n} a^i a^{-i} \\
&= n^{\log_b a} (\log_b n \, + \, 1) = \Theta(n^{\log_b a} \log n),
\end{aligned}$$

Combining this with (2) and the assumption of (B), to within constant factor bounds we have

$$\begin{aligned}
T(n) &= \sum_{i=0}^{\log_b n} a^i f(n/b^i) + O(n^{\log_b a}) = \sum_{i=0}^{\log_b n} a^i (n/b^i)^{\log_b a} + O(n^{\log_b a}) \\
&= \Theta(n^{\log_b a} \log n) + O(n^{\log_b a}) = \Theta(n^{\log_b a} \log n).
\end{aligned}$$

Case (C). The lower bound is immediate, because $f(n)$ is a term of the sum (2). For the upper bound, we will use the smoothness condition. This condition is satisfied by $f(n) = n^{\log_b a + \varepsilon}$ for any $\varepsilon > 0$ with $c = b^{-\varepsilon} < 1$:

$$af(n/b) = a(n/b)^{\log_b a + \varepsilon} = an^{\log_b a + \varepsilon} b^{-\log_b a} b^{-\varepsilon} = f(n) b^{-\varepsilon}.$$

In this case, we have $a^i f(n/b^i) \leq c^i f(n)$ (easy induction on i using the smoothness condition), therefore

$$T(n) = \sum_{i=0}^{\log_b n} a^i f(n/b^i) + O(n^{\log_b a}) \leq \sum_{i=0}^{\log_b n} c^i f(n) + O(n^{\log_b a})$$

$$\leq f(n) \sum_{i=0}^{\infty} c^i + O(n^{\log_b a}) = f(n) \frac{1}{1-c} + O(n^{\log_b a}) = O(f(n)).$$

$\square$